Math 522 Exam 5 Solutions

1. Let $c_0 = 0, c_1 = 1, c_2 = 1, c_3 = 2, c_4 = 5, \ldots$ denote the Catalan numbers. Let $s_n = \sum_{i=0}^{n} c_i$ denote the sum of the first n Catalan numbers. Find a generating function $S(x)$ for the sequence $\{s_n\}$.

   Let $A(x) = \frac{1}{1-x} = \sum_{n\geq 0} 1x^n$. Recall that $C(x) = \frac{1-\sqrt{1-4x}}{2}$ is the generating function for the Catalan numbers. We now calculate $A(x)C(x) = \sum_{n\geq 0} \left(\sum_{i=0}^{n} c_i a_{n-i}\right) x^n = \sum_{n\geq 0} \left(\sum_{i=0}^{n} c_i\right) x^n = \sum_{n\geq 0} s_n x^n$. Hence the desired generating function is $S(x) = A(x)C(x) = \frac{1-\sqrt{1-4x}}{2(1-x)}$.

2. Let $A(x) = \frac{5}{(1-2x)(1+3x)}$ be the generating function for $\sum_{n\geq 0} a_n x^n$. Find a closed form for a_n.

   We begin with partial fractions: $A(x) = \frac{S}{1-2x} + \frac{T}{1+3x} = \frac{S(1+3x)+T(1-2x)}{(1-2x)(1+3x)}$. We get the system $S + T = 5, 3S - 2T = 0$, with solution $S = 2, T = 3$. Hence $A(x) = 2\sum_{n\geq 0} 2^n x^n + 3\sum n \geq 0 (-3)^n x^n = \sum_{n\geq 0}(2 \cdot 2^n + 3(-3)^n)x^n$. Hence $a_n = (2 \cdot 2^n + 3(-3)^n)$, which can be simplified as $a_n = 2^{n+1} - (-3)^{n+1}$.